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Invariants at fixed and arbitrary energy. A unified geometric approach.

Kjell Rosquist†§ and Giuseppe Pucacco‡

† Department of Physics, Stockholm University, Box 6730, 113 85 Stockholm, Sweden
and

RGGR group, Chimie-Physique, Université Libre de Bruxelles, Brussels, Belgium

‡ Physics Department, University of Rome 'Tor Vergata', Via della Ricerca Scientifica 1, I-00133 Rome, Italy

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Abstract. Invariants at arbitrary and fixed energy (strongly and weakly conserved quantities) for two-dimensional Hamiltonian systems are treated in a unified way. This is achieved by utilizing the Jacobi metric geometrization of the dynamics. Using Killing tensors we obtain an integrability condition for quadratic invariants which involves an arbitrary analytic function $S(z)$. For invariants at arbitrary energy the function $S(z)$ is a second-degree polynomial with real second derivative. The integrability condition then reduces to Darboux's condition for quadratic invariants at arbitrary energy. The four types of classical quadratic invariants for positive-definite two-dimensional Hamiltonians are shown to correspond to certain conformal transformations. We derive the explicit relation between invariants in the physical and Jacobi time gauges. In this way knowledge about the invariant in the physical time gauge enables one to write down directly the components of the corresponding Killing tensor for the Jacobi metric. We also discuss the possibility of searching for linear and quadratic invariants at fixed energy and its connection with the problem of the third integral in galactic dynamics. In our approach linear and quadratic invariants at fixed energy can be found by solving a linear ordinary differential equation of the first or second degree, respectively.

1. Introduction

The quest for integrable systems is still one of the main areas of interest in classical dynamics both *per se* and for the applications in the related fields of celestial mechanics, accelerator physics and so on. In galactic dynamics a long-standing, and to this day unresolved, issue concerns the celebrated third integral. Briefly, the observed motion of stars in the galaxy has a certain regularity which indicates the existence of a third invariant in addition to the energy and angular momentum. This picture is confirmed by numerical simulations. In modern terminology, the motion is non-chaotic, in contrast to what one would expect for a generic potential. A particularly striking numerical experiment was made by Hénon and Heiles [1] in 1964 using a model potential now known as the Hénon–Heiles potential. They showed that the motion in that potential is regular with invariant tori up to a certain energy value E_0 . At this energy the invariant tori abruptly begin to break up and the motion becomes chaotic in increasingly large parts of the phase space. One conclusion is that there can be no invariant which commutes with the Hamiltonian. On the other hand, the very

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high degree of regularity at low energies, $E < E_0$, seems to indicate the existence of an additional invariant at those energies. Such an invariant, if it exists, must necessarily depend on the energy with some kind of singular behaviour at $E = E_0$. However, no such invariant is known for any value of the energy. One of the purposes of this paper is to discuss the possibility of finding invariants at fixed energy values.

An essential aspect of integrability is therefore to check for the existence of integrals of the motion in addition to the energy, namely phase-space functions invariant with respect to the phase flow. The search for additional invariants has therefore been actively pursued for more than a century (Bertrand in 1852 [2]; Darboux in 1901 [3]; for a complete review of the matter see Hietarinta [4]). A lesser-known aspect of the question is the relation between the invariants that most frequently appear in the applications, namely polynomials in the momenta, and Killing tensors of the configurational manifold of the dynamical system. This relation is in the form of a straightforward correspondence if the Jacobi approach to the geometrization of the dynamics of a conservative system is adopted.

The framework of the Jacobi geometrical formulation of the dynamics offers several conceptual and technical aids that give the possibility of shedding new light on a number of aspects of integrability. From the technical point of view, it allows the use of the powerful tools of Riemannian geometry on the configurational manifold of the corresponding natural Jacobi formulation of the dynamical system. From the conceptual point of view, it treats invariants at fixed energy (the so-called configurational invariants of Hall [5]) on the same footing as the invariants at arbitrary energy, thus realizing a unified geometric approach.

One aim of the present paper is to demonstrate the fruitfulness of the geometric approach by providing the generalization of Darboux's conditions to include the configurational invariants. In the particular but fundamental case of two degrees of freedom, the Killing equations for second-rank Killing tensors can be solved in full generality, resulting in all the known cases of integrability at arbitrary energy, both the classical ones already obtained by Darboux [3] and those recently found to exhaust all the possibilities in two dimensions (Dorizzi *et al* [6]).

The substantially larger family of solutions of the Killing equation in the fixed energy case allows one to find large new classes of constrained integrable systems, with a backwards procedure going from the knowledge of the form of the invariants to the assessment of the structure of the potentials admitting constrained integrability. Applying a new prescription for transforming Hamiltonian symmetries between different time gauges, the relation between the invariants in the physical and Jacobi time gauges is explicitly determined. The relation between the invariants also provides a shortcut to calculate the Killing tensor from the knowledge of the invariant in the physical time gauge.

2. Geometric representation of the dynamics

The geometrization of the dynamics in terms of Jacobi geometry has been known for about a century, and can be found in some text books [7–9]. In spite of this it is not widely known or used, and we therefore outline the main ideas in this section to make the paper self-contained. Most Hamiltonian systems of physical interest have a geometric kinetic energy part, that is to say that the kinetic energy is a non-degenerate quadratic form in the momenta p_α

$$T = \frac{1}{2} h^{\alpha\beta} p_\alpha p_\beta \quad (1)$$

where $h_{\alpha\beta}$ is a function of the configuration variables q^α . The Hamiltonian itself then has the form

$$H = T + V(q) = E . \tag{2}$$

The independent variable t is often, but not always, the time. For simplicity we shall refer to the independent variable as the time in this paper. For any given energy E of the system we can use the Hamiltonian

$$\mathcal{H} = H - E \tag{3}$$

to represent the dynamics, provided that we impose the constraint

$$\mathcal{H} = 0 . \tag{4}$$

For any such zero-energy Hamiltonian we can reparametrize the system by introducing a new time variable t_N defined by the relation

$$dt = \mathcal{N}(p, q) dt_N \tag{5}$$

together with a redefined Hamiltonian

$$\mathcal{H}_N = \mathcal{N}(p, q)\mathcal{H} = \mathcal{N}T + \mathcal{N}(V - E) = 0 . \tag{6}$$

The new Hamiltonian will then give the same equations of motion on the constraint surface $\mathcal{H}_N = 0$ (see, e.g., [10]). We shall use the term *lapse function* for $\mathcal{N}(p, q)$ which defines the independent variable gauge. This usage is borrowed from applications in general relativity where the lapse gives the rate of physical time change relative to coordinate time (see, e.g., [11]). The lapse function can be taken to be any non-zero function on the phase space.

The dynamics of the system can be represented in a purely geometric formulation by exploiting the reparametrization freedom. The passage to the geometric representation is accomplished by defining a new time variable t_J (*Jacobi time*) by the lapse choice

$$\mathcal{N} = \mathcal{N}_J = [2(E - V)]^{-1} . \tag{7}$$

Note that for a positive-definite kinetic energy $E - V$ is always non-negative in the physically allowed region. The corresponding Hamiltonian has the form $\mathcal{H}_N = 2(E - V)^{-1}T - \frac{1}{2}$. Thus \mathcal{H}_N has a constant potential energy which can be subtracted without affecting the equations of motion. This leads to the *Jacobi Hamiltonian*, H_J , defined by

$$H_J = \frac{1}{2}\mathcal{N}_J h^{\alpha\beta} p_\alpha p_\beta = \frac{1}{2}J^{\alpha\beta} p_\alpha p_\beta \tag{8}$$

where $J^{\alpha\beta}$ is the *Jacobi metric* of the system. This defines a geometry which contains all information about the dynamics. The dynamical metric is given in covariant form by

$$J_{\alpha\beta} = 2(E - V)h_{\alpha\beta} . \tag{9}$$

The Jacobi metric is therefore conformally related to the original kinetic metric $h_{\alpha\beta}$. It follows that the dynamics of the system (2) is equivalent to geodesic motion in the Jacobi geometry†

$$ds^2 = J_{\alpha\beta} dq^\alpha dq^\beta . \tag{10}$$

Note that the Jacobi geometry depends on the energy parameter E . This means that in general the Jacobi geometries corresponding to different energy surfaces are inequivalent. The complete geometric representation of the dynamics of the system is therefore given

† It may be that there are other geometries which can also be used to represent the dynamics. In that case the dynamical geometry defined according to the prescription given here is not unique. For example an inequivalent dynamical geometry may sometimes be obtained by performing a suitable canonical transformation in some time gauge (see, e.g., [12]).

by the geodesics of a one-parameter family of geometries. Because of the form of the conformal factor in the Jacobi geometry (9) the geometric representation is only valid locally in configuration space at points where $V \neq E$. In general the equation $V = E$ defines a non-empty energy surface in configuration space for a given value of E . Exceptions occur if, e.g., $V > 0$ throughout the configuration space and $E \leq 0$. In this paper we are only concerned with the local existence of invariants. Therefore the failure of the Jacobi geometry to represent the global dynamics does not affect our analysis.

3. Positive-definite two-dimensional Hamiltonian systems

To study two-dimensional systems it is very helpful to use variables which are null (light-like) with respect to the dynamical metric. Such variables are naturally adapted to the action of the conformal group which plays an essential role for two-dimensional systems. The indefinite (Lorentzian) signature case was discussed by Rosquist and Uggla [13]. The null variables for that case are real and the conformal group can be parametrized by two arbitrary real functions of one variable. Our approach is to use the method of [13] to treat the case of a positive-definite dynamical metric.

We consider a general two-dimensional positive-definite dynamical metric written in the manifestly conformally flat form

$$ds^2 = 2G(x, y)(dx^2 + dy^2). \quad (11)$$

We seek a condition on G which guarantees the existence of a second-rank Killing tensor. To that end we follow [13] as closely as possible and introduce null variables which in the positive-definite case are complex

$$\begin{aligned} z &= x + iy \\ \bar{z} &= x - iy. \end{aligned} \quad (12)$$

Here and in the following a bar is used to denote complex conjugation. The metric then becomes

$$ds^2 = 2G(z, \bar{z})dzd\bar{z}. \quad (13)$$

Note that although our notation is similar to that in [13] the variables z and \bar{z} are complex in the positive-definite case as opposed to the Lorentzian case where they are real.

When doing calculations it is convenient to employ either an orthonormal frame $\omega^{\hat{i}}$ using hatted upper case Latin indices ($\hat{I}, \hat{J}, \hat{K}, \dots = \hat{1}, \hat{2}$) in terms of which the metric is written as

$$ds^2 = (\omega^{\hat{1}})^2 + (\omega^{\hat{2}})^2 \quad (14)$$

or a complex null frame Ω^I using upper case Latin indices ($I, J, K, \dots = 1, 2$) with the metric written as

$$ds^2 = 2\Omega^1\Omega^2. \quad (15)$$

The frame components are given by

$$\begin{aligned} \omega^{\hat{1}} &= (2G)^{1/2}dx & \Omega^1 &= G^{1/2}dz \\ \omega^{\hat{2}} &= (2G)^{1/2}dy & \Omega^2 &= G^{1/2}d\bar{z} \end{aligned} \quad (16)$$

The two frames are related by $\Omega^I = e^I_j \omega^j$ or $\omega^j = e_I^j \Omega^I$ where e_I^j is the transposed matrix inverse of e^I_j . The transformation matrix is given by

$$(e^I_j) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \quad (e_I^j) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}. \tag{17}$$

The simplest symmetry of the geodesic equations is known as a Killing vector field (see, e.g., [14]). A second-rank Killing tensor is a symmetric tensor $K_{IJ} = K_{(IJ)}$ satisfying the equation

$$K_{(IJ;K)} = 0. \tag{18}$$

The existence of such a tensor is equivalent to the existence of a polynomial invariant of the second-degree $K^{IJ} p_I p_J$ for the geodesic equations. Killing tensors and second-degree invariants are the natural generalizations of Killing vectors K^I and the corresponding first-degree invariants $K^I p_I$. Any Killing vector gives rise to a Killing tensor $K_{(I} K_{J)}$. Such a Killing tensor is said to be reducible. An important property of the second-rank Killing tensor equations is that they can be decomposed in conformal (traceless) and trace parts according to

$$\begin{aligned} P_{(IJ;K)} - \frac{1}{2} h_{(IJ} P^L_{K);L} &= 0 \\ K_{;I} &= -P^J_{I;J} \end{aligned} \tag{19}$$

where the Killing tensor itself is decomposed in a conformal part P_{IJ} and the trace $K = K^I_I$ according to

$$K_{IJ} = P_{IJ} + \frac{1}{2} K h_{IJ}. \tag{20}$$

Referring to the vector $P_I = P^J_{I;J}$ as the conformal current, it follows from (19) that for any given conformal Killing tensor the equation for the trace can be solved if the integrability condition

$$P_{[I;J]} = 0 \tag{21}$$

is satisfied. The procedure to solve the Killing tensor equations is therefore to first solve the conformal Killing tensor equations and then check if the integrability condition (21) for the trace can be satisfied. In fact it turns out that the conformal equations can easily be solved leaving the integrability condition as the remaining equation to study.

Let $K_{\hat{i}\hat{j}}$ be the components of a Killing tensor with respect to the orthonormal frame $\omega^{\hat{i}}$. Then the null frame components are given by $K_{IJ} = e_I^{\hat{M}} e_J^{\hat{N}} K_{\hat{M}\hat{N}}$ which gives

$$\begin{aligned} K_{11} &= \frac{1}{2}(K_{\hat{1}\hat{1}} - K_{\hat{2}\hat{2}}) - iK_{\hat{1}\hat{2}} \\ K_{22} &= \frac{1}{2}(K_{\hat{1}\hat{1}} - K_{\hat{2}\hat{2}}) + iK_{\hat{1}\hat{2}} \\ K_{12} &= \frac{K}{2} = K_{\hat{1}\hat{1}} + K_{\hat{2}\hat{2}} \end{aligned} \tag{22}$$

where $K = K^I_I$ is the trace. Since the orthonormal components $K_{\hat{i}\hat{j}}$ are by definition real it follows that $K_{22} = \overline{K_{11}}$. Therefore we can represent the null frame components of the conformal part of the Killing tensor by a single complex function $S(z, \bar{z})$ according to (cf [13])

$$(K_{MN}) = \begin{pmatrix} \bar{S}G & \frac{K}{2} \\ \frac{K}{2} & SG \end{pmatrix} \tag{23}$$

the trace K being a real function. The conformal Killing tensor equations are formally identical to those obtained in the Lorentzian case [13]

$$C_{111} = G^{1/2} \bar{S}_{,z} = 0 \quad C_{222} = G^{1/2} S_{,\bar{z}} = 0. \quad (24)$$

It follows that K_{MN} is a conformal Killing tensor precisely if S is a function of z alone. One way of stating this result is that the equations (24) for the conformal Killing tensor coincide with the Cauchy–Riemann equations for certain rescaled linear combinations of the tracefree parts of the Killing tensor. This result was implicit in [15]. Thus the conformal Killing tensor has the form

$$(P_{MN}) = \begin{pmatrix} \bar{S}(\bar{z})G & 0 \\ 0 & S(z)G \end{pmatrix}. \quad (25)$$

The trace equations can then be written in the form

$$\begin{aligned} K_{,z} &= -2\bar{S}(\bar{z})G_{,\bar{z}} - G\bar{S}'(\bar{z}) \\ K_{,\bar{z}} &= -2S(z)G_{,z} - GS'(z) \end{aligned} \quad (26)$$

the derivative $K_{,\bar{z}}$ being given by taking the complex conjugate of the above equation.

As in the Lorentzian case there is an infinite-dimensional family of conformal Killing tensors in this case parametrized by the single complex analytic function $S(z)$. Following the procedure in [13] we now write down the integrability condition for the conformal current. This is exactly what we need to guarantee the existence of a Killing tensor. In the positive-definite case the integrability condition becomes

$$\begin{aligned} P_{[1;2]} &= -G^{-1}G_{,zz}S(z) + G^{-1}G_{,\bar{z}\bar{z}}\bar{S}(\bar{z}) - \frac{3}{2}G^{-1}G_{,z}S'(z) + \frac{3}{2}G^{-1}G_{,\bar{z}}\bar{S}'(\bar{z}) \\ &\quad - \frac{1}{2}S''(z) + \frac{1}{2}\bar{S}''(\bar{z}) = 0 \end{aligned} \quad (27)$$

or in a more compact form

$$P_{[1;2]} = -i \operatorname{Im} \{ 2G^{-1}G_{,zz}S(z) + 3G^{-1}G_{,z}S'(z) + S''(z) \} = 0 \quad (28)$$

where Im denotes the imaginary part. As in [13] we use a conformal transformation to standardize the frame and coordinate representation of the conformal Killing tensor. To that end we introduce the new null frame $\tilde{\Omega}^1 = B\Omega^1$, $\tilde{\Omega}^2 = B^{-1}\Omega^2$ and a new complex coordinate by the transformation $w = H(z)$ with inverse $z = F(w)$. By choosing $B = (\bar{F}'(\bar{w})/F'(w))^{1/2}$ we ensure that the new frame $\tilde{\Omega}^I$ has the coordinate representation $\tilde{\Omega}^1 = \tilde{G}^{1/2}dw$, $\tilde{\Omega}^2 = \tilde{G}^{1/2}d\bar{w}$ where $\tilde{G} = |F'(w)|^2G$ is the new metric conformal factor, $ds^2 = 2\tilde{G}dw d\bar{w}$. It follows that the conformal Killing tensor components in the new frame are given by:

$$\begin{aligned} \tilde{P}_{11} &= B^{-2}P_{11} = [\bar{H}'(\bar{z})]^2 \bar{S}(\bar{z}) \tilde{G} \\ \tilde{P}_{22} &= B^2P_{22} = [H'(z)]^2 S(z) \tilde{G}. \end{aligned} \quad (29)$$

The coordinates are standardized by choosing a conformal transformation function $H(z)$ which satisfies $[H'(z)]^{-2} = S(z)$ implying that $\tilde{P}_{11} = \tilde{P}_{22} = \tilde{G}$. Note that $\det(P_{IJ})$ is always positive. Therefore there is only one type of Killing tensors for positive-definite Hamiltonians, unlike the situation in the Lorentzian case (see [13]). For a standardized conformal Killing tensor the integrability (27) condition simplifies to

$$\tilde{G}_{,ww} = \tilde{G}_{,\bar{w}\bar{w}}. \quad (30)$$

Writing $w = X + iY$ the solution of this equation is

$$\tilde{G} = Q_1(X) + Q_2(Y). \quad (31)$$

This is the usual form of the potential in separable coordinates.

The trace equation (see equation (19)) can then be written

$$\begin{aligned} K_{,x} &= -2\tilde{G}_{,x} \\ K_{,y} &= 2\tilde{G}_{,y} \end{aligned} \tag{32}$$

with the solution

$$K = -2Q_1(X) + 2Q_2(Y). \tag{33}$$

The integrability condition (27) can be interpreted as follows. Given any analytic function $S(z)$ there is a family of potentials (the solutions of (27)) which is integrable at zero energy. For future reference we note that the original potential is given in terms of the standardized potential by the relation

$$G = |S(z)|^{-1} \tilde{G}. \tag{34}$$

We wish to express the invariant $I_J = K^{MN} p_M p_N$ in terms of the coordinate momentum components. The subscript J is used to distinguish the invariant in the Jacobi time gauge from the invariant I in the physical time gauge (see section 6). By (16) the null frame components are related to the complex coordinate components by $(p_1, p_2) = G^{-1/2}(p_z, p_{\bar{z}})$. The invariant can then be written in complex coordinates as

$$I_J = S(z) p_z^2 + \bar{S}(\bar{z}) p_{\bar{z}}^2 + G^{-1} K p_z p_{\bar{z}}. \tag{35}$$

Using the relation

$$\begin{aligned} p_z &= \frac{1}{2}(p_x - ip_y) \\ p_{\bar{z}} &= \frac{1}{2}(p_x + ip_y) \end{aligned} \tag{36}$$

implied by (12) we obtain the invariant in real coordinates as

$$I_J = \frac{1}{2} \text{Re}(S)(p_x^2 - p_y^2) + \text{Im}(S)p_x p_y + \frac{1}{4} G^{-1} K(p_x^2 + p_y^2). \tag{37}$$

Note that the last term is equal to KH_J .

4. Arbitrary energy invariants

A common situation is that one is interested in invariants which are valid for arbitrary values of the energy. Invariants of that type arise if the integrability condition itself is independent of the energy. This happens precisely if $\text{Im}\{S''(z)\} = 0$. Since $S''(z)$ is an analytic function it follows that $S''(z)$ is a real constant and therefore $S(z)$ must be a second-degree polynomial

$$S(z) = W(z) := az^2 + \beta z + \gamma \tag{38}$$

where a is a real constant and β and γ are complex constants. The integrability condition (27) then simplifies to

$$\text{Im}\{2G_{,zz}S(z) + 3G_{,z}S'(z)\} = 0. \tag{39}$$

This is simply Darboux's condition for quadratic constants of the motion [3, 4] written in complex variables. The general integrability condition (27) therefore generalizes Darboux's condition to include also quadratic invariants at fixed energy.

For a given analytic function $S(z)$ the potential may be written as $G(X, Y) = |F'(w)|^{-2}(Q_1(X) + Q_2(Y))$. In the arbitrary energy case we have $S(z) = W(z) = [H'(z)]^{-2}$.

The equation $H'(z) = [W(z)]^{-1/2}$ can then be integrated resulting in (without loss of generality we may put $a = 1$ if $a \neq 0$ since (39) is invariant under a real scaling of S)

$$w = H(z) = \begin{cases} \log(\sqrt{W} + z + \beta/2) + \text{constant} & (a = 1) \\ 2\beta^{-1}\sqrt{\beta z + \gamma} + \text{constant} & (a = 0, \beta \neq 0) \\ \gamma^{-1/2}z + \text{constant} & (a = 0, \beta = 0). \end{cases} \quad (40)$$

Inverting these relations we must distinguish between the four cases

$$\begin{array}{lll} \text{(i)} & a = 1 & \Delta := \sqrt{\beta^2/4 - a\gamma} \neq 0 \\ \text{(ii)} & a = 1 & \Delta = 0 \\ \text{(iii)} & a = 0 & \beta \neq 0 \\ \text{(iv)} & a = 0 & \beta = 0 \end{array} \quad (41)$$

giving rise to the following explicit formulae for $F(w)$:

$$z = F(w) = \begin{cases} \Delta \cosh(w - w_0) - \beta/2 & \text{(i)} \\ e^{w-w_0} - \beta/2 & \text{(ii)} \\ \frac{1}{4}\beta(w - w_0)^2 - \gamma/\beta & \text{(iii)} \\ \gamma^{1/2}w - w_0. & \text{(iv)} \end{cases} \quad (42)$$

where w_0 is an arbitrary constant. We are primarily interested in proper conformal transformation, i.e. $S(z) \neq 1$. Therefore we are free to perform translations and rotations in z and w to simplify formulae. In particular we may put w_0 equal to zero by a translation of the origin in the w -plane. We can also rotate the z -plane to transform Δ to a positive real number in case (i). Likewise β may be assumed real in case (iii) and γ can be taken as real and positive in case (iv). The argument we used to set $a = 1$ can then be used to set $\beta = 4$ and $\gamma = 1$ for convenience in cases (iii) and (iv) respectively. We also translate the z -origin obtaining finally

$$F(w) = \begin{cases} \Delta \cosh w & \text{(i)} \\ e^w & \text{(ii)} \\ w^2 & \text{(iii)} \\ w & \text{(iv)} \end{cases} \quad (43)$$

where Δ is now to be understood as a positive real number. Differentiation of the expressions (43) yields

$$F'(w) = \begin{cases} \Delta \sinh w & \text{(i)} \\ e^w & \text{(ii)} \\ 2w & \text{(iii)} \\ 1 & \text{(iv)} \end{cases} \quad (44)$$

implying that the conformal transformation factors take the forms

$$|S(z)| = |F'(w)|^2 = \begin{cases} \Delta^2(\sinh^2 X + \sin^2 Y) = \sqrt{(r^2 + \Delta^2)^2 - 4\Delta^2x^2} & \text{(i)} \\ e^{2X} = r^2 & \text{(ii)} \\ 4(X^2 + Y^2) = 4r & \text{(iii)} \\ 1 & \text{(iv)} \end{cases} \quad (45)$$

where $r := \sqrt{x^2 + y^2}$.

In case (i) the conformal coordinate transformation is given by

$$\begin{aligned} x &= \Delta \cosh X \cos Y \\ y &= \Delta \sinh X \sin Y . \end{aligned} \quad (46)$$

This is the transformation used to define the Stäckel potential [16] which was also discussed by Darboux [3]. Case (i) therefore gives Stäckel's classical integrable potential. From (45) and (34) we find that the potential is given by the well known formula [4]

$$G(x, y) = \frac{Q_1(X(x, y)) + Q_2(Y(x, y))}{\Delta^2[\sinh^2 X(x, y) + \sin^2 Y(x, y)]} = \frac{A(\xi(x, y)) + B(\eta(x, y))}{\xi(x, y) + \eta(x, y)} \quad (47)$$

where

$$\begin{aligned} \xi(x, y) &:= 2\Delta^2 \sinh^2 X = r^2 - \Delta^2 + \sqrt{(r^2 + \Delta^2)^2 - 4\Delta^2x^2} \\ \eta(x, y) &:= 2\Delta^2 \sin^2 Y = -r^2 + \Delta^2 + \sqrt{(r^2 + \Delta^2)^2 - 4\Delta^2x^2} . \end{aligned} \quad (48)$$

and the functions A and B are arbitrary functions of their arguments. The case (ii) conformal transformation is given by

$$\begin{aligned} x &= e^X \cos Y \\ y &= e^X \sin Y . \end{aligned} \quad (49)$$

In particular, it follows that $Y = \arctan(y/x) =: \phi$ is a polar angle. Using (45) it then follows that the potential can be written as:

$$G(x, y) = A(r) + r^{-2}B(\phi) \quad (50)$$

where A and B are arbitrary functions. This is again a classical integrable case [4] known as the Eddington potential. Case (iii) is characterized by the conformal transformation

$$\begin{aligned} x &= X^2 - Y^2 \\ y &= 2XY . \end{aligned} \quad (51)$$

Using (45) this gives immediately the likewise well known classical integrable potential [4]

$$G(x, y) = [A(r + x) + B(r - x)]/r \quad (52)$$

where again A and B are arbitrary functions. Finally in case (iv) the conformal transformation factor is unity and so the potential is simply given by the explicitly separate form

$$G(x, y) = A(x) + B(y) \quad (53)$$

in terms of the arbitrary functions A and B . This completes the list of the four classical cases. The three nontrivial conformal transformations giving rise to systems with a second linear or quadratic invariant are given in table 1.

Table 1. Conformal transformation functions giving rise to systems with a second invariant which is linear or quadratic in the momenta.

Case	Conformal transformation $w = H(z)$	Conformal Killing tensor component $S(z) = [H'(z)]^{-2}$	Associated potential
(i)	$\ln z \pm \sqrt{z^2 - \Delta^2}$	$z^2 - \Delta^2$	(47) (Stäckel)
(ii)	$\ln z$	z^2	(50) (Eddington)
(iii)	\sqrt{z}	$4z$	(52)

5. The Killing vector subcase

Although Killing vectors correspond to reducible second-rank Killing tensors it is nevertheless worthwhile to give a separate treatment of that subcase. As will be shown below the integrability condition for Killing vectors is a first order differential equation. In some applications it can be advantageous to investigate this simpler case before tackling the second-rank Killing tensors. A Killing vector can be described by a function $Z(z, \bar{z})$ according to (cf [13])

$$\begin{aligned}
 K_1 &= G^{1/2} \bar{Z} \\
 K_2 &= G^{1/2} Z.
 \end{aligned}
 \tag{54}$$

The Killing vector equations then become

$$\begin{aligned}
 \overline{K_{(1;1)}} &= K_{(2;2)} = Z_{,\bar{z}} = 0 \\
 K_{(1;2)} &= \frac{1}{2} (Z_{,z} + \bar{Z}_{,\bar{z}} + G^{-1} G_{,z} Z + G^{-1} G_{,\bar{z}} \bar{Z}) = 0.
 \end{aligned}
 \tag{55}$$

This shows that Z depends only on z and is therefore a complex analytic function. It follows that the remaining Killing vector equation (56) reduces to a form analogous to the integrability condition for the second-rank Killing tensors (27). It can also be written in the form

$$K_{(1;2)} = G^{-1} \operatorname{Re}\{GZ'(z) + G_{,z}Z(z)\} = G^{-1} \operatorname{Re}\{(GZ)_{,z}\} = 0.
 \tag{57}$$

Any Killing vector gives rise to a second-rank Killing tensor given by

$$K_{IJ} = K_I K_J + c h_{IJ}
 \tag{58}$$

where c is an arbitrary (real) constant. The conformal Killing tensor components then become

$$\begin{aligned}
 P_{11} &= K_{11} = (K_1)^2 = G \bar{Z}^2 \\
 P_{22} &= K_{22} = (K_2)^2 = G Z^2
 \end{aligned}
 \tag{59}$$

Referring to (23) it follows that the conformal Killing tensor is determined by the analytic function

$$S = Z^2.
 \tag{60}$$

As in the second-rank case, invariance at arbitrary energy involves a further restriction coming from the requirement that (56) should be invariant with respect to energy redefinitions. This means that we must impose the condition

$$\operatorname{Re}[Z'(z)] = 0
 \tag{61}$$

leading to the simplified Killing vector equation

$$G_{,z}Z + G_{,\bar{z}}\bar{Z} = 0. \tag{62}$$

where

$$Z(z) = U(z) := ibz + \delta \tag{63}$$

and b and δ are real and complex constants respectively. It follows that the corresponding second-rank conformal Killing tensor is determined by

$$S(z) = [U(z)]^2 = -b^2z^2 + 2ib\delta z + \delta^2. \tag{64}$$

Thus $S(z)$ is an even square for a reducible Killing tensor so the condition $\Delta^2 = \beta^2/4 - a\gamma = 0$ is always satisfied.

6. Transforming Hamiltonian symmetries between different time gauges

In this section we address the problem of how an invariant is affected when we transform from one time gauge to another. Suppose we are given a Hamiltonian H with a second invariant I so that $\{I, H\} = 0$. In another time gauge the Hamiltonian is given by $\mathcal{H}_N = \mathcal{N}\mathcal{H} = \mathcal{N}(H - E)$ where $\mathcal{N} = dt/d\tilde{t}$. Of particular interest for the purposes of this paper is the transformation between the physical and Jacobi time gauges. In that case, going from the physical time gauge to the Jacobi time gauge, $dt = \mathcal{N}d\tilde{t}$, we have $\mathcal{N} = |2V|^{-1}$. Since the Poisson bracket

$$\{I, \mathcal{H}_N\} = \mathcal{H}\{I, \mathcal{N}\} \tag{65}$$

is in general non-vanishing off the zero-energy surface $\mathcal{H} = 0$, the original invariant does not have a vanishing Poisson bracket with the Hamiltonian in the new time gauge. Borrowing usage from Dirac's theory of constrained Hamiltonians we may say that the invariant is only weakly conserved in the new time gauge. For a linear invariant, e.g. the Killing vector case, one can always choose variables such that $I = p_{\tilde{x}}$ and \tilde{x} is a cyclic variable in the Hamiltonian. Then for the Jacobi time gauge, \mathcal{N} depends only on the remaining variable and hence $\{I, \mathcal{H}_N\} = 0$ implying that I is strongly conserved in the new time gauge. Suppose now that we have a two-dimensional system with a second invariant given in the physical time gauge. Then, the system is integrable and we can express the lapse explicitly as a function $\mathcal{N} = f(t)$ of the physical time t . Integrating the relation $d\tilde{t} = dt/f(t)$ then gives the new time as an explicit function of the old time at least up to a quadrature. In principle, the solutions can consequently always be transformed to another time gauge. One therefore expects that the integrability properties of a dynamical system are independent of the time gauge. In order to exploit fully the geometrical formulation of mechanics it is desirable to find a corresponding invariant which is strongly conserved in the Jacobi time gauge. As we shall see this is actually possible at least in the cases considered in this paper.

To find the invariant I_N in the new time gauge it is convenient to use an ansatz of the form

$$I_N = I + R\mathcal{H}_N. \tag{66}$$

We look for a condition on R which guarantees that I_N is a constant of the motion for \mathcal{H}_N . To that end we form the Poisson bracket

$$\{I_N, \mathcal{H}_N\} = \mathcal{N}\mathcal{H}_N \left[\{R, \mathcal{H}\} - \{I, \mathcal{N}^{-1}\} - \mathcal{H}_N\{R, \mathcal{N}^{-1}\} \right]. \tag{67}$$

Requiring this expression to vanish off the zero-energy surface yields the condition

$$\{R, \mathcal{H}\} = \{I, \mathcal{N}^{-1}\} + \mathcal{H}_N\{R, \mathcal{N}^{-1}\}. \tag{68}$$

This is a necessary and sufficient condition for $I_{\mathcal{N}}$ to be a constant of the motion for $\mathcal{H}_{\mathcal{N}}$. In general a solution to this partial differential equation would be difficult to find. For our purposes, however, it turns out that we can actually find a solution by a simple procedure which we now outline. Suppose now that I is a quadratic invariant. Then if \mathcal{N} is a function on the configuration space, the bracket $\{I, \mathcal{N}^{-1}\}$ is a linear function of the momenta. Suppose we find a function R on the configuration space which satisfies $\{R, \mathcal{H}\} = \{I, \mathcal{N}^{-1}\}$. Then since $\mathcal{H}_{\mathcal{N}}\{R, \mathcal{N}^{-1}\} = 0$ we have a solution of (68). In this way the procedure to find the invariant in the new time gauge is reduced to calculating the Poisson bracket $\{I, \mathcal{N}^{-1}\}$ and finding a function whose time derivative coincides with that bracket. Although we do not know under which conditions this procedure works it does work in the cases considered in this paper. To summarize, we first calculate the function $\{I, \mathcal{N}^{-1}\}$ and check whether it can be expressed as a total time derivative of some function R . If in addition $\{R, \mathcal{N}^{-1}\} = 0$ then R is the required function which satisfies (68).

7. The relation between the quadratic invariants in the physical and Jacobi time gauges

We wish to see how the invariant (37) appears in the physical time gauge in terms of the conformal function $S(z)$ and the trace K . To that end we use the procedure outlined in section 6 going 'backwards' from the Jacobi time gauge to the physical time gauge. Referring to section 2 the starting point is now the Jacobi Hamiltonian $H = H_J = \frac{1}{4}G^{-1}(p_x^2 + p_y^2)$ considered at the energy value $\frac{1}{2}$ so $\mathcal{H} = H_J - 1/2$. The transformation from the Jacobi time to the physical time is then given by $dt_J = \mathcal{N}dt$ where $\mathcal{N} = 2G$. According to (66) we write the physical time invariant as $I = I_J + R\mathcal{N}\mathcal{H}$. Using the prescription in section 6, R should satisfy the equation $\{I_J, \mathcal{N}^{-1}\} = \{R, H_J\}$. Calculating the left-hand side $\{I_J, \mathcal{N}^{-1}\}$ with I_J given by the expression (37) gives

$$\{I_J, \mathcal{N}^{-1}\} = G^{-3}(SGG_{,z} + \frac{1}{2}KG_{,\bar{z}})p_z + G^{-3}(\bar{S}GG_{,\bar{z}} + \frac{1}{2}KG_{,z})p_{\bar{z}}. \quad (69)$$

Assuming that R is a function on the configuration space, the right-hand side is given by

$$\{R, H_J\} = G^{-1}(R_{,\bar{z}}p_z + R_{,z}p_{\bar{z}}). \quad (70)$$

Comparing equations (69) and (70) then leads to

$$\begin{aligned} R_{,z} &= \bar{S}G^{-1}G_{,\bar{z}} + \frac{1}{2}KG^{-2}G_{,z} \\ R_{,\bar{z}} &= SG^{-1}G_{,z} + \frac{1}{2}KG^{-2}G_{,\bar{z}}. \end{aligned} \quad (71)$$

At this point it is convenient to introduce a function $Q = R + \frac{1}{2}KG^{-1}$. Equations (71) then reduce to

$$\begin{aligned} Q_{,z} &= -\frac{1}{2}\bar{S}_{,\bar{z}} \\ Q_{,\bar{z}} &= -\frac{1}{2}S_{,z} \end{aligned} \quad (72)$$

where we have used the trace equation (26). The integrability condition for this equation coincides with the condition for integrability at arbitrary energy, $\text{Im } S''(z) = 0$. Using (38) we then find that the solution of (72) is given by

$$Q = -az\bar{z} - \frac{1}{2}\bar{\beta}z - \frac{1}{2}\beta\bar{z} + Q_0 \quad (73)$$

where Q_0 is a (real) integration constant.

Collecting our results we find that the expression for the invariant in the physical time gauge is

$$I = I_{\text{conf}} + \frac{1}{2}Q(p_x^2 + p_y^2) + \frac{K}{2} - GQ \tag{74}$$

where

$$I_{\text{conf}} = P^{MN} p_M p_N = \frac{1}{2} \text{Re}(S)(p_x^2 - p_y^2) + \text{Im}(S)p_x p_y \tag{75}$$

is the conformal part of the invariant. It follows that this part is the same in both time gauges. It is also seen that Q can be identified with the trace of the physical invariant with respect to the physical metric δ_{ab} where a and b are coordinate indices taking the values x and y . The relation between the invariants provides a shortcut to calculate the Killing tensor from knowledge of the physical invariant. We shall outline this procedure and then illustrate with an example. Let the physical invariant be given by an expression of the form

$$I = Q^{ab}(x, y)p_a p_b + f(x, y) \tag{76}$$

The physical Hamiltonian is given by

$$H = \frac{1}{2}\delta^{ab} p_a p_b + V(x, y) \tag{77}$$

We can read off the function Q by $Q = \delta^{ab} Q_{ab}$. Taking the conformal part and identifying with (75) we obtain

$$S = 2(P^{xx} + iP^{xy}) \tag{78}$$

where $P^{ab} = Q^{ab} - \frac{1}{2}Q\delta^{ab}$ and $P^{yy} = -P^{xx}$. Finally we obtain the Killing tensor trace as $K = 2(f + GQ)$ putting $G = E - V$.

Let us now illustrate the above results by an example. We take the Kepler potential $V = -\mu/r$ where $r = \sqrt{x^2 + y^2}$ with its well known non-trivial quadratic invariant, the Laplace–Runge–Lenz vector (see, e.g., [17]) with components

$$\begin{aligned} L_1 &:= e_x = \frac{1}{\mu}(xp_y^2 - yp_x p_y) - \frac{x}{r} \\ L_2 &:= e_y = \frac{1}{\mu}(yp_x^2 - xp_x p_y) - \frac{y}{r}. \end{aligned} \tag{79}$$

The homogeneous and inhomogeneous parts of the invariants L_1 and L_2 are consequently given by

$$Q_{(1)}^{ab} = \begin{pmatrix} 0 & -y/(2\mu) \\ -y/(2\mu) & x/\mu \end{pmatrix} \quad f_{(1)} = -x/r \tag{80}$$

and

$$Q_{(2)}^{ab} = \begin{pmatrix} y/\mu & -x/(2\mu) \\ -x/(2\mu) & 0 \end{pmatrix} \quad f_{(2)} = -y/r. \tag{81}$$

It follows that the conformal Killing tensor components are given by:

$$\begin{aligned} P_{(1)}^{xx} &= -P_{(1)}^{yy} = -x/(2\mu) & P_{(1)}^{xy} &= -y/(2\mu) \\ P_{(2)}^{xx} &= -P_{(2)}^{yy} = y/(2\mu) & P_{(2)}^{xy} &= -x/(2\mu) \end{aligned} \tag{82}$$

while the traces are given by

$$K_{(1)} = 2(E/\mu)x \quad K_{(2)} = 2(E/\mu)y. \tag{83}$$

8. Applications to integrability at fixed energy

An important lesson to be learned from the present work is that integrability at fixed energy and arbitrary energy (weak and strong conservation laws) are just two aspects of the same phenomenon. In particular, a fixed energy invariant in the physical time gauge corresponds to an arbitrary energy invariant if the system is geometrized by going to the Jacobi time gauge. To illustrate how this works in practice we give a few examples in section 8.1 of fixed energy invariants beginning with the Kepler potential. Surprisingly, we find two apparently unknown linear invariants at zero energy for the Kepler problem. It is remarkable that it is still possible to discover new properties of such a simple and well known system. This is in fact a sign of the power of the geometric formulation of dynamical systems. In section 8.2 we discuss how conformal transformations can be used to generate systems which are integrable at fixed energy.

8.1. Degeneracy of the Laplace–Runge–Lenz vector at zero energy

Consider now the Kepler potential, $V = -\mu/r$, using polar coordinates defined by $x = r \cos \phi$, $y = r \sin \phi$. In this case we have one linear invariant at arbitrary energy, the angular momentum p_ϕ . We are interested in finding out whether there exists another linear invariant at some fixed energy value. If this is the case the Jacobi metric (9) has two Killing vectors. However, a two-dimensional space with two Killing vectors must necessarily also have a third Killing vector ([18], theorem 8.15) and the geometry is then a space of constant curvature. This can easily be determined by computing the scalar curvature

$${}^{(2)}R = 2G^{-2}G_{,z\bar{z}} - 2G^{-3}G_{,z}G_{,\bar{z}} \quad (84)$$

of the metric (13) and checking if it is constant. For the Kepler potential we have $G = E + \mu(z\bar{z})^{-1/2}$ and the scalar curvature becomes

$${}^{(2)}R = -\frac{E\mu}{2[\mu + E(z\bar{z})^{1/2}]^3}. \quad (85)$$

This shows that the Jacobi geometry has constant curvature only if $E = 0$ and then the geometry is actually flat. Of course the flatness of the Jacobi geometry in this case also follows directly from the form of the metric since G is then a product of functions of z and \bar{z} . The two extra Killing vector fields can immediately be written down if we introduce Cartesian coordinates, (X, Y) , for the Jacobi geometry by the transformation $w = X + iY = \sqrt{2z}$ leading to the manifestly Euclidean form

$$ds_J^2 = 4\mu(dX^2 + dY^2). \quad (86)$$

The relation to the original coordinates is given by the parabolic transformation

$$x = \frac{1}{2}(X^2 - Y^2) \quad y = XY. \quad (87)$$

This transformation does not have a unique inverse. However, in the region $X = \text{Re}(w) > 0$ we may select the inverse transformation to be

$$X = \sqrt{r+x} \quad Y = (\text{sgn } y)\sqrt{r-x}. \quad (88)$$

It follows that the Killing vector fields are $K_{(1)} = \partial/\partial X$, $K_{(2)} = \partial/\partial Y$ and $K_{(3)} = -Y\partial/\partial X + X\partial/\partial Y = 2\partial/\partial\phi$. The two extra Killing vector fields are thus the translation

symmetries $K_{(1)}$ and $K_{(2)}$. The corresponding invariants are

$$\begin{aligned} I_1 &:= p_x = (r+x)^{1/2} p_x + (\text{sgn } y)(r-x)^{1/2} p_y \\ &= (2r)^{1/2} \cos(\phi/2) p_r - (2/r)^{1/2} \sin(\phi/2) p_\phi \\ I_2 &:= p_y = -(\text{sgn } y)(r-x)^{1/2} p_x + (r+x)^{1/2} p_y \\ &= (2r)^{1/2} \sin(\phi/2) p_r + (2/r)^{1/2} \cos(\phi/2) p_\phi . \end{aligned} \tag{89}$$

These two invariants are related by the quadratic formula $I_1^2 + I_2^2 = 8\mu H_J^0$ where $H_J^0 = -(2V)^{-1}T = \frac{1}{2}$ is the Jacobi Hamiltonian for the zero-energy system. In terms of the physical Hamiltonian, the corresponding relation is $I_1^2 + I_2^2 = 4\mu(1 - H/V) = 4\mu$. In fact here we have the key to the physical interpretation of I_1 and I_2 . To see this let us introduce an invariant ϕ_0 at zero energy by the relations

$$I_1 = -2\mu^{1/2} \sin(\phi_0/2) \quad I_2 = 2\mu^{1/2} \cos(\phi_0/2) . \tag{90}$$

Now using (89) to solve for the radial momentum yields $p_r = \sqrt{2\mu/r} \sin[(\phi - \phi_0)/2]$. Inserting this value into the Hamiltonian constraint $H = 0$ and solving for r gives the familiar relation

$$r = \frac{p_\phi^2/\mu}{1 + \cos(\phi - \phi_0)} . \tag{91}$$

This shows that ϕ_0 is simply the angular integration constant. It follows that changing the value of I_1 (or I_2) only affects the parametrization of the orbit while leaving the orbit itself invariant. From this point of view these invariants are gauge symmetries of the Kepler system at zero energy.

The invariants I_1 and I_2 are in fact closely related to the components of the Laplace–Runge–Lenz vector. Expressing those components as

$$e_x = e \cos \phi_0 \quad e_y = e \sin \phi_0 \tag{92}$$

where e is the eccentricity ($e = 1$ at zero energy) and comparing with (90) it is evident that

$$L_1 = -(2\mu)^{-1} I_1^2 + 1 \quad L_2 = -(2\mu)^{-1} I_1 I_2 . \tag{93}$$

This can also be seen directly by calculating for example I_1^2 from the expression given in (89) with result

$$I_1^2 = 2(r+x)H + 2\mu(1 - L_1) \tag{94}$$

where we have used (79). Now solving for L_1 at zero energy gives again the first of relations (93).

Relations (93) imply that the second-rank Killing tensors corresponding to the components of the Laplace–Runge–Lenz vector are reducible at zero energy. The invariants in the Jacobi time gauge can be found from relations (82), (83), (78), (37). This gives

$$\begin{aligned} J_1 &= -(2\mu)^{-1} x(p_x^2 - p_y^2) - \mu^{-1} y p_x p_y \\ J_2 &= (2\mu)^{-1} y(p_x^2 - p_y^2) - \mu^{-1} x p_x p_y \end{aligned} \tag{95}$$

where J_1 and J_2 are the Jacobi invariants corresponding to L_1 and L_2 , respectively. From (89) we then find relations

$$\begin{aligned} J_1 &= -(2\mu)^{-1} I_1^2 + 2H_J \\ J_2 &= -(2\mu)^{-1} I_1 I_2 . \end{aligned} \tag{96}$$

The reducibility of the Killing tensors $K_{(1)}^{MN}$ and $K_{(2)}^{MN}$ corresponding to J_1 and J_2 is therefore expressed by the formulae

$$\begin{aligned} K_{(1)}^{MN} &= -(2\mu)^{-1} K_{(1)}^M K_{(1)}^N + J^{MN} \\ K_{(2)}^{MN} &= -(2\mu)^{-1} K_{(1)}^M K_{(2)}^N. \end{aligned} \quad (97)$$

We also wish to understand the commutation relations for the fixed energy invariants in the physical time gauge. To facilitate the calculations the physical Hamiltonian is first expressed in terms of the Jacobi Hamiltonian by

$$\mathcal{H} = T + V - E = (V - E)(1 - 2H_J) \quad (98)$$

where we have used $T = 2(E - V)H_J$. We can now exploit the fact that the invariant commutes with the Jacobi Hamiltonian to obtain

$$\{I, H\} = \{I, \mathcal{H}\} = (1 - 2H_J)\{I, V\} = \mathcal{H}(V - E)^{-1}\{I, V\}. \quad (99)$$

This relation shows that we only need to compute the Poisson bracket with the potential. It also follows that the bracket $\{I, H\}$ in general depends linearly on \mathcal{H} .

Using (99) to calculate the brackets for the Kepler invariants we find

$$\begin{aligned} \{I_1, H\} &= r^{-1}(r+x)^{1/2}H = \sqrt{2}r^{-1/2}\cos(\phi/2)H \\ \{I_2, H\} &= -(\operatorname{sgn} y)r^{-1}(r-x)^{1/2}H = -\sqrt{2}r^{-1/2}\sin(\phi/2)H. \end{aligned} \quad (100)$$

8.2. Some other examples of integrability at fixed energy

As discussed for example by Hietarinta [4], conformal transformations provide links between physically different systems which are integrable at some fixed energy. In particular if the original potential \tilde{V} is separable, $\tilde{V} = Q_1(X) + Q_2(Y)$, then the transformed system has the potential

$$V = |H'(z)|^2\{Q_1(\operatorname{Re}(H(z))) + Q_2(\operatorname{Im}(H(z))) - E\} \quad (101)$$

where $z = z + iy$ and $H(z) = X + iY$. One of the results of the present work is that we have identified those conformal transformations for which the new potential in this situation is actually integrable at arbitrary energy (see table 1). Conversely, if the conformal transformation is not contained in table 1 then the resulting potential does not have a linear or quadratic invariant at arbitrary energies. Hietarinta considered conformal transformations of the forms $H(z) = z^m$ (with $m = -1, -2, \frac{1}{2}, 2$), e^z and $\ln z$. Note in particular that of these $H(z) = z^{1/2}$ and $H(z) = \ln z$ produce systems which are integrable at arbitrary energy if the original potential is separable. In this section we give some further examples of simple systems which are integrable at a fixed energy.

Consider first the polynomial function $S(z) = iz^2$. This is the simplest polynomial which gives a potential which is not automatically integrable at arbitrary energy. The corresponding conformal transformation is given by $w = H(z) = 2^{-1/2}(1 - i)\ln z$ or in terms of the real variables

$$\begin{aligned} X &= \frac{1}{\sqrt{2}}(r + \theta) \\ Y &= \frac{1}{\sqrt{2}}(-r + \theta). \end{aligned} \quad (102)$$

From the relation (34) it then follows that the potential given by

$$G = r^{-2}\{A(r + \theta) + B(r - \theta)\} \quad (103)$$

is integrable at zero energy for arbitrary functions A and B . For functions containing linear and quadratic terms the potential takes the form

$$G = a_1 r^{-1} + a_2 \theta r^{-1} + a_3 r^{-2} + a_4 \theta r^{-2} + a_5 (1 + \theta^2 r^{-2}) \tag{104}$$

where the a_i ($i = 1, \dots, 5$) are arbitrary constants.

As another example we take a function of the form $S(z) = z^k$ where $k \neq 0, 1, 2$ is a real constant. This leads to $w = H(z) = m^{-1} z^m$ where $m = -k/2 + 1 \neq 0, \frac{1}{2}, 1$. The conformal transformation can then be written

$$\begin{aligned} X &= m^{-1} r^m \cos(m\theta) \\ Y &= m^{-1} r^m \sin(m\theta) . \end{aligned} \tag{105}$$

The corresponding potential is

$$G = r^{-k} [A(X) + B(Y)] . \tag{106}$$

Choosing for example $A(X) = a_1 m^s X^s$ and $B(Y) = a_2 m^s Y^s$ we have

$$G = r^{-2+m(s+2)} [a_1 \cos^s(m\theta) + a_2 \sin^s(m\theta)] . \tag{107}$$

Specializing to the case $m = 2$ while keeping s arbitrary and using $\cos(2\theta) = (x^2 - y^2)r^{-2}$, $\sin(2\theta) = 2xyr^{-2}$ yields finally the potential

$$G = r^2 [a_1 (x^2 - y^2)^s + a_2 2^s x^s y^s] \tag{108}$$

which is therefore integrable at zero energy.

9. Concluding remarks

It was shown in section 3 that conformal transformations given by analytic functions $H(z)$ for which the condition $\text{Im}\{S''(z)\} = 0$ with $S(z) = [H'(z)]^{-2}$ is satisfied give rise to the classical potentials which admit quadratic (or linear) second invariants at arbitrary energies. Conformal transformations which do not satisfy that condition give potentials which admit quadratic second invariants only at a fixed energy. Our approach unifies the description of quadratic invariants at arbitrary and fixed energies. In particular the integrability condition (27) is valid for both types of invariants. It reduces to Darboux's classical condition for arbitrary energy invariants when $\text{Im}\{S''(z)\} = 0$. Whether a unification can also be achieved for third-degree invariants or higher remains an open problem.

An intriguing aspect of the integrability condition (27) is the possibility that for a given potential function $G = E - V$ there could exist a family of solutions $S(z, E)$ with a continuous dependence on the energy. This would lead to new families of integrable potentials with energy-dependent quadratic invariants. At this point we cannot exclude the existence of such solutions of the integrability condition. A related result was given by Hietarinta [4], who showed that the potential x/y is in fact integrable by energy-dependent invariants which are certain transcendental functions of the momenta.

For a given potential V the integrability condition (27) with $G = E - V$ can be used to determine energy values for which there exists an invariant of at most second degree. This involves solving a linear differential equation of second order. For linear invariants it is sufficient to solve the linear equation (56). We consider this possibility to test for linear and quadratic integrability at fixed energy to be an important application of the geometric approach to Hamiltonian dynamics. Another approach is to look for conditions involving curvature invariants such as the technique used in section 8.1 to find cases with additional invariants. In fact, conditions involving curvature invariants for (1+1)-dimensional models

have recently been found (using two different approaches) for potentials which do not require the manifest linear invariant present in the Kepler problem [19,20]. There are indications that at least the approach of [19] can be generalized to incorporate quadratic invariants as well. It is of considerable interest to develop these techniques and use them to look for fixed energy invariants of physically interesting models such as the Hénon–Heiles potential and others.

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